

# Stability of Kähler-Ricci flow

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*Abstract* We prove the convergence of Kähler-Ricci flow with some small initial curvature conditions. As applications, we discuss the convergence of Kähler-Ricci flow when the complex structure varies on a Kähler-Einstein manifold.

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## 1 Introduction

### 1.1 The motivation

In [17], R. Hamilton introduced the famous Ricci flow which deforms any Riemannian metric in the direction of negative Ricci curvature. If the Ricci flow exists globally, this will lead to the existence of some canonical geometric structure (either Einstein metrics or some non-trivial solitons). Unfortunately, the flow usually will develop singularities in finite time. One key issue is to study the formation of singularities over finite time [19]. According to R. Hamilton, the singularity can be divided roughly into two types: the fast forming one (Type I) and the slow forming one (Type II). From the PDE point of view, the Type I singularity is more “gentle” where the curvature is controlled by  $\frac{1}{T-t}$ . We will focus our discussions on the “fast forming” singularities now.

Let  $g$  be any Riemannian metric and we start the Ricci flow from  $g$ . Suppose the maximal existence time is  $T(g) < \infty$ . Suppose further that this is a Type I singularity. One intriguing question one may ask is: for any metric in a small neighborhood of  $g$ , will the maximal existence time be  $T(g)$  also? Will that be Type I? Note that Einstein metrics with positive scalar curvature, or Ricci solitons (either stable or shrinking), develop exactly finite time Type I singularities. The question above can be simplified to

**Question 1.1.** [5] *Let  $g$  be a stable or shrinking Ricci soliton. Does there exist a small neighborhood where Ricci flow will still develop only Type I singularities? Will the Ricci flow asymptotically converge to some Ricci soliton?*

This is a very interesting question. Unfortunately, the answer in full generality should be negative: Even if we assume a metric  $g$  is  $C^\infty$  close to an Einstein metric with positive scalar curvature, it is not clearly at all whether the flow will only develop Type I singularities, and

whether the flow will converge to an Einstein metric after proper re-scaling. Nonetheless, it is an intriguing question to find a set of suitable geometric conditions so that this question has an affirmative answer. This type of problem is called linear stabilities and there are lots of research on this topic (cf. [24], [11], [16], [36] etc).

One way to approach this problem is to associate certain functional to Ricci flow. In general setting, the first choice of course will be Perelman's entropy functional. Unfortunately, Perelman's entropy functional depends on the blowing up time  $T$  and we have no idea how  $T$  will vary when we vary initial metric.

It is now clear why we should study this important problem on Kähler manifolds first. In any compact manifold, for the Ricci flow initiating from any Kähler metric, the blowing up time is not only a priori determined, but more importantly it can be explicitly computed in terms of the first Chern class and the Kähler class. Therefore, Question 1.1 will be highly interesting and feasible with present technology when restricted to the Kähler setting. A more modest question one may ask is

**Question 1.2.** [5] *Let  $(M, g, J)$  be a Kähler-Einstein manifold. Does there exist a small neighborhood of the Kähler-Einstein metric such that the Kähler-Ricci flow will converge to a Kähler-Einstein metric?*

If we don't perturb the complex structure, the answer is yes due to the unpublished work of G. Perelman [25], as well as Tian-Zhu [33]. If we indeed perturb the complex structure, then the answer is not clear. LeBrun-Simanca gives some results on the related problem in [21]. An example of G. Tian [32] and S. K. Donaldson [13] shows that the Mukai-Umemura manifold  $X$  admits a Kähler-Einstein metric, but some small deformation of  $X$  has no Kähler-Einstein metrics.

The question we want to ask is of more general than Question 1.2.

**Question 1.3.** [5] *Let  $(M, g, J)$  be an "almost Kähler-Einstein manifold" in some natural sense. Will the Kähler-Ricci flow converge to a Kähler-Einstein metric (regardless of the complex structure)?*

This is the project we want to study in this paper. We will give some stability results of Kähler-Ricci flow with respect to the deformation of the underlying complex structures and prove some new convergence results with small energy conditions.

## 1.2 On the deformation of complex structures

Let  $(M, [\omega])$  be a polarized compact Kähler manifold with  $[\omega] = 2\pi c_1(M) > 0$  (the first Chern class) in this paper. One of the main theorem we prove in this paper is:

**Theorem 1.4.** *Let  $(M, g_{KE}, J_{KE})$  be a Kähler-Einstein manifold with  $c_1(M) > 0$ , and no non-zero holomorphic vector fields. For any Kähler metric  $\omega_g \in 2\pi c_1(M) \cap H^{1,1}(M, J)$  with possibly different complex structure  $J$  satisfying*

$$\|(g, J) - (g_{KE}, J_{KE})\|_{C^2} \leq \epsilon, \quad (1.1)$$

*for sufficiently small  $\epsilon(g_{KE}, J_{KE}) > 0$ , the Kähler-Ricci flow with the initial metric  $(\omega_g, J)$  will converge exponentially fast to a Kähler-Einstein metric.*

**Remark 1.5.** *Theorem 1.4 can be derived from a combination of N. Koiso's results (cf. Proposition 10.1 in [20]) and Perelman or Tian-Zhu's results [33]. More precisely, N. Koiso proved that if  $M$  has no non-zero holomorphic vector fields, for any one-parameter complex deformation  $J_t$  of complex structure  $J_{KE}$ , there exists a sequence of Einstein metrics  $g_t$  which are Kähler metrics compatible with  $J_t$ .*

Next we consider the case that  $(M, J)$  has non-zero holomorphic vector fields. As in [7], we need to assume  $(M, J)$  is pre-stable, which means that the complex structure doesn't jump under the action of diffeomorphism group of  $M$  (cf. Definition 3.2). Under this assumption, we have the following result:

**Theorem 1.6.** *Let  $(M, g_{KE}, J_{KE})$  be a Kähler-Einstein manifold with  $c_1(M) > 0$ . For any Kähler metric  $\omega_g \in 2\pi c_1(M) \cap H^{1,1}(M, J)$  with possibly different complex structure  $J$  satisfying the following conditions:*

1.  $(M, J)$  is pre-stable;
2.  $([\omega_g], J)$  has vanishing Futaki invariant;
3.  $\|(g, J) - (g_{KE}, J_{KE})\|_{C^2} \leq \epsilon$ , for sufficiently small  $\epsilon(g_{KE}, J_{KE}) > 0$ ;

*the Kähler-Ricci flow with the initial metric  $(\omega_g, J)$  will converge exponentially fast to a Kähler-Einstein metric.*

**Remark 1.7.** *Lebrun-Simanca proved the existence of constant-scalar-curvature Kähler metrics for a deformation of complex structures on a Kähler manifold (cf. Theorem 5 in [21]). They assume that the Futaki invariant is non-degenerate, which says that the linearization of the Futaki invariant in the direction of the Kähler class is injective. Note that the Futaki invariant of a Kähler-Einstein metric is never non-degenerate (cf. [21]). Our case seems to be complimentary to the case considered in [21].*

The proof of Theorem 1.4 and 1.6 follows directly from Theorem 1.8 and 1.10 respectively (cf. Section 5, 6). The idea of the proof follows from our previous paper [7], and we will discuss the details in Section 1.3.

### 1.3 On the convergence of Kähler-Ricci flow

In our previous paper [7], we proved some convergence theorems for the Kähler-Ricci flow with certain initial energy and curvature conditions. Here we will refine those arguments and prove the following type of stability results :

**Theorem 1.8.** *Let  $(M, J)$  be a Kähler manifold with  $c_1(M) > 0$ . For any  $\gamma, \Lambda > 0$ , there exists a small positive constant  $\epsilon(\gamma, \Lambda) > 0$  such that for any metric  $g$  in the subspace of Kähler metrics*

$$\{ \omega_g \in 2\pi c_1(M) \mid \lambda_1(\omega_g) > 1 + \gamma, \quad |Rm|(\omega_g) \leq \Lambda, \quad Ca(\omega_g) \leq \epsilon \}, \quad (1.2)$$

*where  $\lambda_1(g)$  is the first eigenvalue of the metric  $\omega_g$  and  $Ca(\omega_g)$  denotes the (normalized) Calabi energy, the Kähler-Ricci flow with the initial metric  $\omega_g$  will converge exponentially fast to a Kähler-Einstein metric.*

**Remark 1.9.** *The assumption (1.2) implies that  $(M, J)$  has no non-zero holomorphic vector fields. Note that we don't assume the existence of Kähler-Einstein metrics on  $(M, J)$ .*

One might tempt to think that condition (1.2) implies the existence of Kähler-Einstein metrics. That might be true (one need to address a potential collapsing issue with (1.2)), except that it is a Kähler-Einstein metric with possibly different complex structures. The difficulty is really created by the fact the space of complex structure modulo diffeomorphisms is not a Hausdorff space.

As in Theorem 1.6, when  $(M, J)$  has non-zero holomorphic vector fields, we need to assume the pre-stable condition.

**Theorem 1.10.** *Let  $(M, J)$  be a Kähler manifold with  $c_1(M) > 0$ . Suppose  $(M, J)$  is pre-stable and the Futaki invariant of the class  $2\pi c_1(M)$  vanishes. For any  $\Lambda > 0$ , there exists  $\epsilon(\Lambda) > 0$  such that for any metric  $g$  with its Kähler form  $\omega_g$  in the subspace of Kähler metrics*

$$\{ \omega_g \in 2\pi c_1(M) \mid |Rm|(\omega_g) \leq \Lambda, \quad Ca(\omega_g) \leq \epsilon \}, \quad (1.3)$$

*the Kähler Ricci flow with the initial metric  $\omega_g$  will converge exponentially fast to a Kähler-Einstein metric.*

**Remark 1.11.** *Under the pre-stable condition on complex structures, Phong-Sturm [26] and Phong-Song-Sturm-Weinkove [28] proved some convergence results of Kähler-Ricci flow with extra curvature conditions. We refer the readers to [27] [23][14] for more recent results on Kähler-Ricci flow.*

This type of stability problems for the Kähler-Ricci flow was initiated in [6] and later in [7] with an assumption on the smallness of energy functional  $E_1$  or  $E_0$ . In this paper, we replace this energy condition by the assumption that the Calabi energy is sufficiently small. Unlike Theorem 1.5 in [7], we don't need any conditions on the potential function of the initial Kähler metric.

The proof of the main theorems is more tricky than that in [7], but the ideas are the same. First by Sprouse's result in [30] the smallness of the Calabi energy implies that the  $L^\infty$  norm of the traceless Ricci curvature is small after a short time (cf. Proposition 4.1), which further implies that the eigenvalue is strictly great than 1 (cf. Lemma 3.1). The eigenvalue estimates can be used to prove the exponential decay of the traceless Ricci curvature for a short time (cf. Theorem 3.6 and 3.7), which implies the full curvature tensor is uniformly bounded for a short time by Yau's estimates (cf. Theorem 3.8). However, the boundedness of the full curvature tensor in turn implies the rough curvature estimates for the next time interval, and we can repeat the previous arguments. Using this "iteration" idea we can actually prove that the full curvature tensor is uniformly bounded for all time.

In our subsequent papers, we will remove the condition on the Futaki invariant and the bound of the full curvature tensor, and give more general results on the relation between the pre-stable condition and the convergence of Kähler-Ricci flow. Recently, Tian-Zhu [34] proved very interesting and much stronger results on stability of Kähler-Ricci flow by using Perelman's  $W$ -functional.

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## 2 Preliminaries

Let  $M$  be a compact Kähler manifold with  $c_1(M) > 0$ . Choose an initial Kähler metric  $g$  with the Kähler form  $\omega_g \in 2\pi c_1(M)$ . By the Hodge theorem, any Kähler form in the same Kähler class can be written as

$$\omega_\varphi = \omega_g + \sqrt{-1}\partial\bar{\partial}\varphi$$

for some real potential function  $\varphi$  on  $M$ . The Kähler-Ricci flow (cf. [17]) on a Kähler manifold  $M$  is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} + g_{i\bar{j}}, \quad \forall i, j = 1, 2, \dots, n. \quad (2.1)$$

It follows that on the level of Kähler potentials, the Kähler-Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega_g^n} + \varphi - h_g, \quad \varphi(0) = 0, \quad (2.2)$$

where  $h_g$  is defined by

$$\text{Ric}(\omega_g) - \omega_g = \sqrt{-1}\partial\bar{\partial}h_g \quad \text{and} \quad \int_M h_g \omega_g^n = 0. \quad (2.3)$$

Let  $\underline{R}$  be the the average of the scalar curvature, which is a constant depending only on the Kähler class and the underlying complex structure. Then the normalized Calabi energy (cf. [1][2]) is defined by

$$Ca(\omega_\varphi) = \frac{1}{V} \int_M (R(\omega_\varphi) - \underline{R})^2 \omega_\varphi^n. \quad (2.4)$$

Since the Kähler metric  $\omega_\varphi$  is in the canonical class, we can check that

$$Ca(\omega_\varphi) = \frac{1}{V} \int_M |Ric(\omega_\varphi) - \omega_\varphi|^2 \omega_\varphi^n.$$

Define the Futaki invariant by

$$f_M([\omega_g], X) = \int_M X(h_g) \omega_g^n,$$

for any holomorphic vector field  $X$  on  $M$ . It is well-known that the Futaki invariant doesn't depend on the particular representative we choose in the Kähler class.

We recall some basic results from our previous papers. First, by the tensor maximum principle we have the following basic lemma:

**Lemma 2.1.** ([6][7]) *Suppose that the curvature of the initial metric satisfies the following condition*

$$\begin{cases} |Rm|(0) & \leq \Lambda, \\ |Ric - \omega|(0) & \leq \epsilon. \end{cases}$$

*there exists a constant  $T(\Lambda) > 0$ , such that we have the following bound for the evolving Kähler metric  $g(t)$  ( $0 \leq t \leq 6T$ )*

$$\begin{cases} |Rm|(t) & \leq 2\Lambda, \\ |Ric - \omega|(t) & \leq 2\epsilon. \end{cases} \quad (2.5)$$

Lemma 2.1 is slightly different from Lemma 12 in [6], but the idea of the proof is the same. We remind the readers that  $T$  doesn't depend on the bound of the traceless Ricci curvature, which is useful for the proofs of the main theorems.

Now we state a parabolic version of Moser iteration argument (cf. [8]).

**Theorem 2.2.** *Suppose the Poincare constant and the Sobolev constant of the evolving Kähler metrics  $g(t)$  are both uniformly bounded by  $\sigma$ , and the scalar curvature  $R(g(t))$  has a uniform lower bound  $\Lambda$ . If a nonnegative function  $u$  satisfying the following inequality*

$$\frac{\partial}{\partial t} u \leq \Delta u + f(t, x)u, \quad \forall a < t < b,$$

*where  $|f|_{L^p(M, g(t))}$  is uniformly bounded by some constant  $c$  for some  $p > \frac{n}{2}$ , then for any  $t \in (a, b)$  and  $\tau \in (0, b - a)$ , we have<sup>1</sup>*

$$u(t) \leq \frac{C(n, \sigma, c, \Lambda)}{\tau^{\frac{n+2}{4}}} \left( \int_{t-\tau}^t \int_M u^2 \omega_\varphi^n \wedge ds \right)^{\frac{1}{2}}.$$

**Remark 2.3.** *Recently, Q. Zhang [38] and R. Ye [37] proved that the Sobolev constant is uniformly bounded along the Kähler-Ricci flow without any assumptions. Here we don't need to use this result.*

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<sup>1</sup>The constant  $C$  can be different at different places with possibly some lower indices. The notation  $C(A, B, \dots)$  means that the constant  $C$  depends only on  $A, B, \dots$

The parabolic Moser iteration theorem is the main tool in the paper. It can be applied to control the pointwise norm of the traceless Ricci curvature, provided that the Calabi energy is sufficiently small. Once we have the bound of the traceless Ricci curvature, we can estimate the potential function and the full curvature tensor along the Kähler-Ricci flow by Yau's estimates:

**Theorem 2.4.** (cf. [7], [9], [35]) *For any positive constants  $\Lambda, B > 0$  and small  $\eta > 0$ , there exists a constant  $C_1$  depending only on  $\Lambda, B, \eta$  such that if the background metric  $\omega$  satisfies*

$$|Rm|(\omega) \leq \Lambda, \quad |Ric(\omega) - \omega| \leq \eta,$$

*and the potential function  $|\varphi(t)|, |\dot{\varphi}(t)| \leq B$ , then*

$$|Rm|(t) \leq C_1(B, \Lambda, \eta).$$

We state the following well-known result on the estimate of the Sobolev constant. The readers are referred to [10][22] for details.

**Theorem 2.5.** *Let  $(M, g)$  be a compact  $m$ -dimensional Riemannian manifold. Suppose  $Ric \geq -\Lambda$ ,  $\text{Vol}(M) \geq \nu > 0$ , and the diameter  $\text{diam}(M) \leq D$ , then there exists a constant  $C_S(\Lambda, \nu, D) > 0$  such that for any function  $f \in C^\infty(M)$ , we have*

$$\left( \int_M |f|^{\frac{2m}{m-2}} dV_g \right)^{\frac{m-2}{m}} \leq C_S \left( \int_M |\nabla f|^2 dV_g + \int_M |f|^2 dV_g \right).$$

### 3 Estimates

In this section, we will prove several results which will be useful in the proof of the main theorems.

#### 3.1 The first eigenvalue of the Laplacian operator

To prove the exponential decay of the traceless Ricci curvature, we need to estimate the first eigenvalue of the evolving Laplacian operator. The calculation of the eigenvalue along the Ricci flow is well-known in literatures(cf. [4] for example).

**Lemma 3.1.** *Let  $\lambda_1(t)$  be the first eigenvalue of the Laplacian operator acting on functions with respect to the metric  $\omega_\varphi$  along the Kähler-Ricci flow.*

1. *If  $|Ric - \omega|(t) \leq \epsilon$  for  $t \in [0, T]$ , then*

$$\lambda_1(t) \geq \lambda_1(0)e^{-3n\epsilon t}, \quad \forall t \in [0, T].$$

2. *If  $|Ric - \omega|(t) \leq \epsilon e^{-\alpha t}$  for some  $\alpha > 0$  and for all  $t \in [0, T]$ , then*

$$\lambda_1(t) \geq \lambda_1(0)e^{-\frac{3n}{\alpha}\epsilon(1-e^{-\alpha t})}, \quad \forall t \in [0, T].$$



*Proof.* Let  $\lambda_1(t)$  be an eigenvalue of  $\Delta_\varphi$  with  $-\Delta_\varphi f(t) = \lambda_1(t)f(t)$ , where  $f(t)$  is a smooth function satisfying the normalization condition

$$\int_M f(t)^2 \omega_\varphi^n = 1.$$

Taking the derivative with respect to  $t$ , we have

$$\int_M \left( 2f \frac{\partial f}{\partial t} + f^2 \Delta_\varphi \frac{\partial \varphi}{\partial t} \right) \omega_\varphi^n = 0. \quad (3.1)$$

Observe that

$$\lambda_1(t) = \int_M |\nabla f|^2 \omega_\varphi^n,$$

we calculate the derivative of  $\lambda_1(t)$

$$\begin{aligned} \frac{d\lambda_1(t)}{dt} &= -\frac{d}{dt} \int_M f \Delta_\varphi f \omega_\varphi^n \\ &= \int_M \left( -\frac{\partial f}{\partial t} \Delta_\varphi f - f \frac{\partial}{\partial t} (\Delta_\varphi f) - f \Delta_\varphi f \Delta_\varphi \frac{\partial \varphi}{\partial t} \right) \omega_\varphi^n \\ &= \int_M \left( -\frac{\partial f}{\partial t} \Delta_\varphi f - f \Delta_\varphi \frac{\partial f}{\partial t} + f \left( \frac{\partial \varphi}{\partial t} \right)_{i\bar{j}} f_{j\bar{i}} - f \Delta_\varphi f \Delta_\varphi \frac{\partial \varphi}{\partial t} \right) \omega_\varphi^n \\ &= \lambda_1 \int_M \left( 2f \frac{\partial f}{\partial t} + f^2 \Delta_\varphi \frac{\partial \varphi}{\partial t} \right) \omega_\varphi^n + \int_M f \left( \frac{\partial \varphi}{\partial t} \right)_{i\bar{j}} f_{j\bar{i}} \omega_\varphi^n. \end{aligned}$$

Applying (3.1), we have that

$$\begin{aligned} \frac{d\lambda_1(t)}{dt} &= \int_M f \left( \frac{\partial \varphi}{\partial t} \right)_{i\bar{j}} f_{j\bar{i}} \omega_\varphi^n \\ &= \int_M \left( (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) + f R_{,\bar{\beta}} f_\beta \right) \omega_\varphi^n \\ &= \int_M (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) \omega_\varphi^n - \int_M R |\nabla f|^2 \omega_\varphi^n + \lambda_1(t) \int_M R f^2 \omega_\varphi^n. \end{aligned} \quad (3.2)$$

The assumption (1) implies

$$Ric(\omega_\varphi) - \omega_\varphi \geq -\epsilon \omega_\varphi, \quad n - n\epsilon \leq R \leq n + n\epsilon.$$

Hence, we get the inequalities

$$\begin{aligned} \frac{d\lambda_1(t)}{dt} &\geq -\epsilon \lambda_1(t) - (n + n\epsilon) \lambda_1(t) + (n - n\epsilon) \lambda_1(t) \\ &\geq -3n\epsilon \lambda_1(t). \end{aligned}$$

The first part of the lemma follows immediately. Similarly we can prove the second part.  $\square$



### 3.2 The pre-stable condition

In this section, we estimate the first eigenvalue when  $M$  has non-zero holomorphic vector fields. Here we follow closely the argument in [7]. First, we recall the following definition in [7]:

**Definition 3.2.** *The complex structure  $J$  of  $M$  is called pre-stable, if no complex structure in the closure of its orbit of diffeomorphism group contains larger (reduced) holomorphic automorphism group.*

**Remark 3.3.** *The "pre-stable" condition was defined in [7], but it is well-known in previous literatures. In the statement of Theorem 1.8 of [6], the first named author used this condition to study the convergence to Kähler-Ricci flow, and in [26] Phong-Sturm defined a stability condition as "condition (B)". These two definitions are essentially the same and we called it "pre-stable" in [7].*

Now we recall some basic facts of the first and second eigenvalues of the Laplacian operator acting on functions. For any smooth function  $f \in C^\infty(M)$  we have

$$\int_M |\nabla \Delta_g f|^2 dV_g \geq \lambda_1 \int_M |\Delta_g f|^2 dV_g, \quad \forall f \in C^\infty(M), \quad (3.3)$$

where  $\lambda_1$  is the first eigenvalue of  $\Delta_g$ . If we assume the function  $f$  is perpendicular to the first eigenspace of  $\Delta_g$  then we get

$$\int_M |\nabla \Delta_g f|^2 dV_g \geq \lambda_2 \int_M |\Delta_g f|^2 dV_g. \quad (3.4)$$

These inequalities can be proved by the eigenvalue decomposition of the function  $f$ . Note that for a Kähler-Einstein manifold  $(M, \omega_{KE})$  with non-zero holomorphic vector fields, it is well-known that the first eigenvalue  $\lambda_1 = 1$  and the first eigenspace, which we denote by  $\eta(M)$ , is isomorphic to the space of holomorphic vector fields.

Now we need the following convergence result of a sequence of Kähler metrics, which is well-known in literature (cf. [26], [31]).

**Proposition 3.4.** *Let  $M$  be a compact Kähler manifold. Let  $(g(t), J(t))$  be any sequence of metrics  $g(t)$  and complex structures  $J(t)$  such that  $g(t)$  is Kähler with respect to  $J(t)$ . Suppose the following is true:*

1. *For some integer  $k \geq 1$ ,  $|\nabla^l Rm|_{g(t)}$  is uniformly bounded for any integer  $l(0 \leq l < k)$ ;*
2. *The injectivity radii  $i(M, g(t))$  are all bounded from below;*
3. *There exist two uniform constant  $c_1$  and  $c_2$  such that  $0 < c_1 \leq \text{Vol}(M, g(t)) \leq c_2$ .*

*Then there exists a subsequence of  $t_j$ , and a sequence of diffeomorphism  $F_j : M \rightarrow M$  such that the pull-back metrics  $\tilde{g}(t_j) = F_j^* g(t_j)$  converge in  $C^{k,\alpha}(\forall \alpha \in (0, 1))$  to a  $C^{k,\alpha}$  metric  $g_\infty$ . The pull-back complex structure tensors  $\tilde{J}(t_j) = F_j^* J(t_j)$  converge in  $C^{k,\alpha}$  to an integral complex structure tensor  $\tilde{J}_\infty$ . Furthermore, the metric  $g_\infty$  is Kähler with respect to the complex structure  $\tilde{J}_\infty$ .*

These being understood, we have the:

**Theorem 3.5.** *Suppose that  $(M, J)$  is pre-stable. For any  $\Lambda_0, \Lambda_1 > 0$ , there exists  $\epsilon > 0$  depending only on  $\Lambda_0$  and  $\Lambda_1$  such that for any metric  $\omega \in 2\pi c_1(M)$ , if*

$$|Ric(\omega) - \omega| \leq \epsilon, \quad |Rm|(\omega) \leq \Lambda_0, \quad |\nabla Rm|(\omega) \leq \Lambda_1, \quad (3.5)$$

*then for any smooth function  $f$  satisfying*

$$\int_M f \omega^n = 0 \quad \text{and} \quad \int_M X(f) \omega^n = 0, \quad \forall X \in \eta(M, J), \quad (3.6)$$

*we have the following*

$$\int_M |\nabla f|^2 \omega^n > (1 + \gamma(\epsilon, \Lambda_0, \Lambda_1)) \int_M |f|^2 \omega^n, \quad (3.7)$$

$$\int_M |\nabla \Delta f|^2 \omega^n > (1 + \gamma(\epsilon, \Lambda_0, \Lambda_1)) \int_M |\Delta f|^2 \omega^n, \quad (3.8)$$

*where  $\gamma > 0$  depends only on  $\epsilon, \Lambda_0$  and  $\Lambda_1$ .*

*Proof.* The inequality (3.7) was proved in [7], and the argument also works for (3.8). For the readers' convenience, we give the details here.

Suppose not, for any positive numbers  $\epsilon_m \rightarrow 0$ , there exists a sequence of Kähler metrics  $\omega_m \in 2\pi c_1(M)$  such that

$$|Ric(\omega_m) - \omega_m| \leq \epsilon_m, \quad |Rm|(\omega_m) \leq \Lambda_0, \quad |\nabla_m Rm|(\omega_m) \leq \Lambda_1, \quad (3.9)$$

where the smooth functions  $f_m$  satisfy

$$\int_M f_m \omega_m^n = 0, \quad \int_M X(f_m) \omega_m^n = 0, \quad \forall X \in \eta(M, J), \quad (3.10)$$

$$\int_M |\nabla_m \Delta_m f_m|^2 \omega_m^n < (1 + \gamma_m) \int_M |\Delta_m f_m|^2 \omega_m^n, \quad (3.11)$$

where  $0 < \gamma_m \rightarrow 0$ . Without loss of generality, we may assume that

$$\int_M |\Delta_m f_m|^2 \omega_m^n = 1, \quad \forall m \in \mathbb{N}, \quad (3.12)$$

which means

$$\int_M |\nabla_m \Delta_m f_m|^2 \omega_m^n \leq 1 + \gamma_m < 2. \quad (3.13)$$

Then,  $f_m$  will converge weakly in  $W^{3,2}$  if  $(M, \omega_m)$  converges. However, according to Proposition 3.4,  $(M, \omega_m, J)$  will converge in  $C^{2,\alpha}(\alpha \in (0, 1))$  to  $(M, \omega_\infty, J_\infty)$ . In fact, by (3.9) the diameters of  $\omega_m$  are uniformly bounded. Note that all the metrics  $\omega_m$  are in the same Kähler class, the volume is fixed. Then by (3.9) again, the injectivity radii are uniformly bounded from below. Therefore, all the conditions of Proposition 3.4 are satisfied.

Note that the complex structure  $J_\infty$  lies in the closure of the orbit of diffeomorphisms, while  $\omega_\infty$  is a Kähler-Einstein metric of  $(M, J_\infty)$ . By the standard deformation theorem in complex structures, we have

$$\dim \operatorname{Aut}_r(M, J) \leq \dim \operatorname{Aut}_r(M, J_\infty).$$

By abusing notation, we can write

$$\operatorname{Aut}_r(M, J) \subset \operatorname{Aut}_r(M, J_\infty).$$

By our assumption of pre-stable of  $(M, J)$ , we have the inequality the other way around. Thus, we have

$$\dim \operatorname{Aut}_r(M, J) = \dim \operatorname{Aut}_r(M, J_\infty), \quad \text{or} \quad \operatorname{Aut}_r(M, J) = \operatorname{Aut}_r(M, J_\infty).$$

Now, let  $f_\infty$  be the  $W^{3,2}$  limit of  $f_m$ , then by (3.12) and (3.13) we have

$$1 \leq |f_\infty|_{W^{3,2}(M, \omega_\infty)} \leq C,$$

and by (3.10) we have

$$\int_M f_\infty \omega_\infty^n = 0, \quad \int_M X(f_\infty) \omega_\infty^n = 0, \quad \forall X \in \eta(M, J_\infty).$$

Thus,  $f_\infty$  is a non-trivial function. Since  $\omega_\infty$  is a Kähler-Einstein metric, we have

$$\int_M \theta_X f_\infty \omega_\infty^n = 0,$$

where

$$\mathcal{L}_X \omega_\infty = \sqrt{-1} \partial \bar{\partial} \theta_X.$$

This implies that  $f_\infty$  is perpendicular to the first eigenspace of  $\Delta_{\omega_\infty}$ . In other words, there is a  $\delta > 0$  such that

$$\int_M |\nabla_\infty \Delta_\infty f_\infty|^2 \omega_\infty^n > (1 + \delta) \int_M |\Delta_\infty f_\infty|^2 \omega_\infty^n = 1 + \delta.$$

However, this contradicts the following fact:

$$\begin{aligned} \int_M |\nabla \Delta_\infty f_\infty|^2 \omega_\infty^n &\leq \lim_{m \rightarrow \infty} \int_M |\nabla \Delta_m f_m|^2 \omega_m^n \\ &\leq \lim_{m \rightarrow \infty} (1 + \gamma_m) \int_M |\Delta_m f_m|^2 \omega_m^n = 1. \end{aligned}$$

The lemma is then proved. □

### 3.3 The exponential decay of traceless Ricci curvature

In this section, we will use the estimates of the first eigenvalue in the previous subsection to prove the exponential decay of traceless Ricci curvature.

**Theorem 3.6.** *Suppose for any time  $t \in [0, T]$ ,*

$$|Rm|(t) \leq \Lambda, \quad |Ric - \omega|(t) \leq H\epsilon, \quad \lambda_1(t) \geq 1 + \gamma > 1.$$

*Then for fixed  $\tau < T$ , there exists  $\epsilon_0(H, \gamma) > 0$  such that if  $\epsilon \in (0, \epsilon_0)$  we have*

$$|Ric - \omega|(t) \leq C_2(\Lambda, \gamma, \tau) H\epsilon e^{-\frac{\gamma}{4}t}, \quad t \in [\tau, T].$$

*Proof.* First, we prove that the Calabi energy decays exponentially

$$Ca(t) \leq e^{-\gamma t} Ca(0), \quad \forall t \in [0, T]. \quad (3.14)$$

In fact, direct calculation shows

$$\begin{aligned} & \frac{d}{dt} Ca(t) \\ &= \frac{1}{V} \int_M \left( 2\Delta_\varphi \frac{\partial \varphi}{\partial t} \frac{\partial}{\partial t} \left( \Delta_\varphi \frac{\partial \varphi}{\partial t} \right) + \left( \Delta_\varphi \frac{\partial \varphi}{\partial t} \right)^3 \right) \omega_\varphi^n \\ &= \frac{1}{V} \int_M \left( -2\Delta_\varphi \frac{\partial \varphi}{\partial t} \left| \nabla \bar{\nabla} \frac{\partial \varphi}{\partial t} \right|^2 - 2 \left| \nabla \Delta_\varphi \frac{\partial \varphi}{\partial t} \right|^2 + 2 \left| \Delta_\varphi \frac{\partial \varphi}{\partial t} \right|^2 + \left( \Delta_\varphi \frac{\partial \varphi}{\partial t} \right)^3 \right) \omega_\varphi^n \end{aligned} \quad (3.15)$$

Combining (3.3) with the assumptions, we have

$$\frac{d}{dt} Ca(t) \leq -(2\gamma - 3nH\epsilon) Ca(t). \quad (3.16)$$

If we choose  $\epsilon$  sufficiently small, the inequality (3.14) follows immediately.

Now applying the parabolic Moser iteration, the estimate of the Calabi energy implies the exponential decay of the traceless Ricci curvature. In fact, since the traceless Ricci tensor  $u = |Ric - \omega|^2(t)$  satisfies the following inequality:

$$\frac{\partial u}{\partial t} \leq \Delta u + c(n)|Rm|u,$$

By the parabolic Moser iteration Theorem 2.2, for  $t \in [\tau, T]$  we have

$$\begin{aligned} |Ric - \omega|^2(t) &\leq \frac{C(\Lambda)}{\tau^{\frac{m+4}{2}}} \left( \int_{t-\tau}^t \int_M |Ric - \omega|^4 \omega_\varphi^n \wedge ds \right)^{\frac{1}{2}} \\ &\leq \frac{H\epsilon C(\Lambda)}{\tau^{\frac{m+4}{2}}} \left( \int_{t-\tau}^t \int_M |Ric - \omega|^2 \omega_\varphi^n \wedge ds \right)^{\frac{1}{2}} \\ &= \frac{H\epsilon C(\Lambda)}{\tau^{\frac{m+4}{2}}} \left( \int_{t-\tau}^t \int_M \left( \Delta_\varphi \frac{\partial \varphi}{\partial t} \right)^2 \omega_\varphi^n \wedge ds \right)^{\frac{1}{2}} \\ &\leq \frac{C(\Lambda) H \sqrt{Ca(0)} \epsilon}{\sqrt{\gamma} \tau^{\frac{m+4}{2}}} e^{-\frac{\gamma}{2}(t-\tau)}. \end{aligned}$$

Here we have used the fact that the Sobolev constant is uniformly bounded since the traceless Ricci curvature is small by the assumption. Note that at the initial time  $Ca(0) \leq H^2\epsilon^2$ , the above inequality implies

$$|Ric - \omega|(t) \leq \frac{C(\Lambda)H\epsilon}{\gamma^{\frac{1}{4}}\tau^{\frac{m+4}{4}}} e^{-\frac{\gamma}{4}(t-\tau)}.$$

The theorem is proved.  $\square$

When  $M$  has non-zero holomorphic vector fields, we need to use the pre-stable condition to get the exponential decay of the Ricci curvature.

**Theorem 3.7.** *Suppose that  $M$  is pre-stable and the Futaki invariant vanishes, and for any time  $t \in [0, T]$ ,*

$$|Rm|(t) \leq \Lambda, \quad |Ric - \omega|(t) \leq H\epsilon.$$

*Then for fixed  $\tau < T$ , there exists  $\epsilon_0(H, \gamma) > 0$  such that if  $\epsilon \in (0, \epsilon_0)$  we have*

$$|Ric - \omega|(t) \leq C_3(\Lambda, \gamma, \tau)H\epsilon e^{-\frac{\gamma}{4}t}, \quad t \in [\tau, T].$$

*Here  $\gamma$  is the constant obtained in Theorem 3.5.*

*Proof.* The argument is essentially the same as the proof of Theorem 3.6. In fact, by Shi's results in [29] all the derivatives of the curvature tensor are bounded after a short time  $t = \frac{\tau}{2}$ , where  $\tau$  is a fixed number as in Theorem 3.6. Therefore the assumption (3.5) in Theorem 3.5 is satisfied for  $t \in [\frac{\tau}{2}, T]$  and we have the inequalities (3.7) and (3.8). Note that the function  $\frac{\partial \varphi}{\partial t} - c(t)(c(t) = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n)$  satisfies the condition (3.6), since the Futaki invariant vanishes

$$f_M(X) = \int_M X \left( \frac{\partial \varphi}{\partial t} \right) = 0, \quad \forall X \in \eta(M).$$

Therefore, we have the inequality

$$\int_M \left| \nabla \Delta_\varphi \frac{\partial \varphi}{\partial t} \right| \omega_\varphi^n \geq (1 + \gamma) \int_M \left| \Delta_\varphi \frac{\partial \varphi}{\partial t} \right| \omega_\varphi^n, \quad t \in [\frac{\tau}{2}, T],$$

where  $\gamma > 0$  is the constant obtained in Theorem 3.5. Then by the same argument in the proof of Theorem 3.6 we have the decay of the Calabi energy

$$Ca(t) \leq e^{-\gamma(t-\frac{\tau}{2})} Ca(\frac{\tau}{2}) \leq H^2\epsilon^2 e^{-\gamma(t-\frac{\tau}{2})}, \quad \forall t \in [\frac{\tau}{2}, T]. \quad (3.17)$$

Applying the parabolic Moser iteration as in the proof of Theorem 3.6, we get the exponential decay of the traceless Ricci curvature. The theorem is established.  $\square$

### 3.4 The curvature tensor

In this section, we use the exponential decay of the traceless Ricci curvature to estimate the curvature tensor along the Kähler-Ricci flow.

**Theorem 3.8.** *Suppose along the Kähler-Ricci flow, the traceless Ricci curvature*

$$|\text{Ric} - \omega|(t) \leq H\epsilon e^{-\alpha t}, \quad t \in [0, T], \quad (3.18)$$

*then the full curvature tensor is uniformly bounded*

$$|Rm|(t) \leq \Lambda_0(H\epsilon, \alpha, \Lambda), \quad t \in [0, T],$$

*where  $\Lambda$  is the bound of the curvature tensor with respect to the background metric  $\omega_g$ .*

**Remark 3.9.** *Phong-Song-Sturm-Weinkove proved a similar result in their paper (cf. Lemma 6 in [27]). However, the proof here is elementary and we don't need to use Perelman's estimates.*

First we recall a useful normalization for the solution  $\varphi(t)$  of the Kähler-Ricci flow. Observe that for any solution  $\varphi(t)$  of Kähler Ricci flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega_g^n} + \varphi - h_g,$$

the function  $\tilde{\varphi}(t) = \varphi(t) + Ce^t$  also satisfies the above equation for any constant  $C$ . Since  $\frac{\partial \tilde{\varphi}}{\partial t}(0) = \frac{\partial \varphi}{\partial t}(0) + C$ , we have  $\tilde{c}(0) = c(0) + C$ , where

$$\tilde{c}(t) = \frac{1}{V} \int_M \frac{\partial \tilde{\varphi}}{\partial t} \omega_\varphi^n, \quad c(t) = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n.$$

Thus we can normalize the solution  $\varphi(t)$  such that the average of  $\dot{\varphi}(0)$  is any given constant.

Recall that in Chen-Tian's paper [8], if the  $K$ -energy is bounded from below, we can normalize the solution such that  $c(t)$  is uniformly bounded for all  $t > 0$ . However, in our case we have no information about the  $K$ -energy. To overcome this difficulty, we estimate  $c(t)$  as follows:

**Lemma 3.10.** *(cf. [8]) Suppose that for  $t \in [0, T]$*

$$\mu_1(t) = \frac{1}{V} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \omega_\varphi^n \leq C_4 e^{-\alpha t},$$

*for some constant  $C_4 > 0$ . Then we can normalize the solution  $\varphi(t)$  so that*

$$c(0) = \frac{1}{V} \int_0^T e^{-t} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \omega_\varphi^n \wedge dt < C_4.$$

*Then for all time  $t \in [0, T]$ , we have the following estimates*

$$0 < c(t) < C_4 e^{-\alpha t}, \quad \int_0^T c(t) dt < \frac{C_4}{\alpha}.$$

*Proof.* A simple calculation yields

$$c'(t) = c(t) - \frac{1}{V} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \omega_\varphi^n.$$

Now we normalize our initial value of  $c(t)$  as

$$c(0) = \frac{1}{V} \int_0^T e^{-t} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \omega_\varphi^n \wedge dt \leq C_4 \int_0^T e^{-(1+\alpha)t} dt \leq \frac{C_4}{1+\alpha} \leq C_4.$$

From the equation for  $c(t)$ , we have

$$(e^{-t}c(t))' = -\mu_1(t)e^{-t}.$$

Thus, we have

$$0 < c(t) = \frac{1}{V} \int_t^T e^{-(\tau-t)} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2(\tau) \omega_\varphi^n \wedge d\tau \leq C_4 e^{-\alpha t}$$

and

$$\int_0^T c(t) dt = C_4 \int_0^T e^{-\alpha t} dt \leq \frac{C_4}{\alpha}.$$

□

*Proof of Theorem 3.8.* We will use Theorem 2.4 to bound the full curvature tensor. It suffices to bound  $\varphi, \frac{\partial \varphi}{\partial t}$  for time  $t \in [0, T]$ . Here we need to normalize  $\varphi(t)$  such that its average  $c(t)$  has good estimates. Note that the normalization in Lemma 3.10 depends on  $T$ , which is different from [8].

By the assumption (3.18) on the Ricci curvature, we have

$$\mu_1(t) \leq 2Ca(t) \leq 2H^2\epsilon^2 e^{-2\alpha t}, \quad t \in [0, T],$$

where we used that  $\lambda_1 > 1$  under the assumption of the main theorems, or by Lemma 4.13 in [7] the first eigenvalue  $\lambda_1(t) \geq \frac{1}{2}$  if we choose  $\epsilon$  sufficiently small. By Lemma 3.10, we can normalize the solution  $\varphi_1(t) = \varphi(t) + G_1 e^t$ , where

$$G_1 = \frac{1}{V} \int_0^T e^{-t} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \omega_\varphi^n \wedge dt + \frac{1}{V} \int_M h_g \omega_g^n,$$

such that

$$0 < c_1(t) = \frac{1}{V} \int_M \frac{\partial \varphi_1}{\partial t} \omega_\varphi^n < 2H^2\epsilon^2 e^{-2\alpha t}, \quad \int_0^T c_1(t) dt \leq \frac{H^2\epsilon^2}{\alpha}.$$

Since at the initial time  $\varphi(0) = 0$  and  $|h_g|_{C^0} \leq C(\sigma)H\epsilon$  by the normalization condition (2.3), we have

$$|\varphi_1|(0) \leq |G_1| \leq \frac{2H^2\epsilon^2}{1+2\alpha} + C(\sigma)H\epsilon. \quad (3.19)$$



Recall that

$$\Delta_\varphi \left( \frac{\partial \varphi_1}{\partial t} - c_1(t) \right) = n - R,$$

we have

$$\left| \frac{\partial \varphi_1}{\partial t} - c_1(t) \right| \leq C(\sigma) H \epsilon e^{-\alpha t}. \quad (3.20)$$

Then (3.19) and (3.20) imply

$$\left| \frac{\partial \varphi_1}{\partial t} \right| \leq \left| \frac{\partial \varphi_1}{\partial t} - c_1(t) \right| + |c_1(t)| \leq 2H^2 \epsilon^2 e^{-2\alpha t} + C(\sigma) H \epsilon e^{-\alpha t},$$

and

$$\begin{aligned} |\varphi_1|(t) &\leq |\varphi_1|(0) + \left| \int_0^t \left( \frac{\partial \varphi_1}{\partial t} - c_1(t) \right) dt \right| + \int_0^t c_1(t) dt \\ &\leq 2H^2 \epsilon^2 + C(\sigma) H \epsilon + \frac{H^2 \epsilon^2}{\alpha} + \frac{C(\sigma)}{\alpha} H \epsilon. \end{aligned}$$

Then by Theorem 2.4 we have the curvature bound

$$|Rm|(t) \leq \Lambda_0(H\epsilon, \alpha, \Lambda).$$

Here we omit the Sobolev constant in the constant  $\Lambda_0$  because the traceless Ricci curvature is very small. □

## 4 Proof of Theorem 1.8

In this section, we follow the idea of our previous paper [7] to prove Theorem 1.8. The proof needs the technical condition that the first eigenvalue of the initial metric is strictly greater than 1, which will be removed in Section 6 by the pre-stable condition.

### 4.1 The traceless Ricci curvature

In this subsection, we will prove that under the assumption of Theorem 1.8 the traceless Ricci curvature is small after a short time. In fact, we have the following proposition:

**Proposition 4.1.** *Given a Kähler metric  $\omega_g$  satisfying the properties*

$$\lambda_1(\omega_g) > 1 + \lambda, \quad |Rm|(\omega_g) \leq \Lambda, \quad Ca(\omega_g) < \epsilon, \quad (4.1)$$

*for some constants  $\lambda, \Lambda > 0$  and sufficiently small  $\epsilon > 0$ . Then for the solution  $\omega(t)$  of the Kähler-Ricci flow with the initial metric  $\omega_g$ , there exists  $T(\Lambda)$  and  $\epsilon_0(\gamma, \Lambda) > 0$  such that if  $\epsilon \in (0, \epsilon_0)$  then at time  $T$  the metric  $\omega(T)$  satisfies*

$$\lambda_1(T) > 1 + \frac{\gamma}{2}, \quad |Rm|(T) \leq 2\Lambda, \quad |Ric - \omega|(T) \leq C\epsilon^{\frac{1}{4}}, \quad (4.2)$$

*for some constant  $C(\Lambda)$ .*

First, we will prove that the Sobolev constant is uniformly bounded if the Calabi energy is small and the Ricci curvature has a lower bound. We will follow an approach taken by [30] and also [9]. Recall that C. Sprouse proved the following lemma in [30]:

**Lemma 4.2.** *Let  $(M, g)$  be a complete  $m$ -dimensional Riemannian manifold with  $Ric \geq (m - 1)k$  ( $k \leq 0$ ). Then for any  $D, \delta > 0$  there exists  $\epsilon = \epsilon(n, k, D, \delta)$  such that if*

$$\sup_x \frac{1}{\text{Vol}(B(x, D))} \int_{B(x, D)} ((m - 1) - Ric_-)_+ dV < \epsilon, \quad (4.3)$$

*then  $(M, g)$  is compact, with  $\text{diam}(M) < \pi + \delta$ . Here  $Ric_-$  denotes the lowest eigenvalue of the Ricci tensor. For any function  $f$  on  $M$ ,  $f_+ = \max\{f(x), 0\}$ .*

**Remark 4.3.** *After rescaling, the conclusion of Lemma 4.2 should be: Suppose  $Ric \geq -\Lambda$ . For any  $D, \delta, a > 0$  there exists  $\epsilon_0 = \epsilon_0(m, D, \delta, \Lambda, a) > 0$  such that if*

$$\sup_x \frac{1}{\text{Vol}(B(x, D))} \int_{B(x, D)} |Ric - a| dV < \epsilon_0, \quad (4.4)$$

*then the diameter of the metric  $\omega$  is bounded by  $\sqrt{\frac{m-1}{a}}(\pi + \delta)$ .*

Now we use Lemma 4.2 to give a uniform upper bound of the Sobolev constant.

**Lemma 4.4.** *Let  $(M, [\omega])$  be a polarized Kähler manifold and  $[\omega]$  is the canonical Kähler class. For any  $\Lambda > 0$  there exists  $\epsilon_0 = \epsilon_0(\Lambda)$  such that if*

$$Ric(\omega) \geq -\Lambda, \quad Ca(\omega) < \epsilon_0, \quad (4.5)$$

*then the Sobolev constant of the metric  $\omega$  is uniformly bounded by some constant  $\sigma = \sigma(\Lambda)$ .*

*Proof.* Let  $D, \delta$  be some fixed number, for example  $D = 100, \delta = 1$ . Then for any  $x \in M$ , we have

$$\begin{aligned} \frac{1}{\text{Vol}(B(x, D))} \int_{B(x, D)} |Ric - \omega| dV &\leq C(n, D, \Lambda) \left( \int_M |Ric - \omega|^2 \omega^n \right)^{\frac{1}{2}} \\ &\leq C(n, D, \Lambda) \epsilon_0^{\frac{1}{2}}, \end{aligned}$$

where we have used the volume comparison theorem. Then by Lemma 4.2 the diameter is bounded by a constant  $C(n)$  if  $\epsilon_0$  is small enough. Since the volume is fixed and  $Ric \geq -\Lambda$ , by Theorem 2.5 the Sobolev constant is uniformly bounded by  $\sigma(\Lambda)$ .  $\square$

Now we return to the proof of Proposition 4.1.

*Proof of Proposition 4.1.* We start with the metric  $\omega_g$  satisfying (4.1). By the maximum principle, there exists  $T(\Lambda) > 0$  such that along the flow the curvature has the estimates

$$|Rm|(t) \leq 2\Lambda, \quad t \in [0, T]. \quad (4.6)$$

Now we show the Calabi energy is also small for  $t \in [0, T]$  :

**Lemma 4.5.** *Along the Kähler-Ricci flow, if  $|Rm|(t) \leq \Lambda, t \in [0, T]$  then the Calabi energy satisfies*

$$Ca(t) \leq e^{C(\Lambda)t} Ca(0), \quad t \in [0, T].$$

*Proof.* By the curvature assumption and (3.15) we get

$$\begin{aligned} \frac{d}{dt} Ca(t) &= \int_M \left( -2\Delta_\varphi \frac{\partial \varphi}{\partial t} \left| \nabla \bar{\nabla} \frac{\partial \varphi}{\partial t} \right|^2 - 2 \left| \nabla \Delta_\varphi \frac{\partial \varphi}{\partial t} \right|^2 + 2 \left| \Delta_\varphi \frac{\partial \varphi}{\partial t} \right|^2 + \left( \Delta_\varphi \frac{\partial \varphi}{\partial t} \right)^3 \right) \omega_\varphi^n \\ &\leq C(\Lambda) Ca(t). \end{aligned} \quad (4.7)$$

The lemma follows immediately.  $\square$

Since the initial Calabi energy is small, it follows from Lemma 4.5 and (4.6) that

$$Ca(t) \leq C(\Lambda, T)\epsilon, \quad t \in [0, T]. \quad (4.8)$$

Then by Lemma 4.4 the Sobolev constant is uniformly bounded for time  $t \in [0, T]$  :

$$C_S(t) \leq \sigma(\Lambda), \quad t \in [0, T]. \quad (4.9)$$

Combining this with the parabolic moser iteration as in the proof of Theorem 3.6, we have

$$|Ric - \omega|(T) \leq C(\Lambda, \sigma, T) \left( \int_{\frac{T}{2}}^T Ca(s) ds \right)^{\frac{1}{4}} \leq C(\Lambda, \sigma, T) \epsilon^{\frac{1}{4}}. \quad (4.10)$$

So it remains to prove  $\lambda_1(T) > 1 + \frac{\gamma}{2}$  provided  $\epsilon$  is sufficiently small. We remind the readers that it cannot be proved by Lemma 3.1 since the traceless Ricci curvature may not be small near  $t = 0$ . However, we have the lemma:

**Lemma 4.6.** *Let  $\lambda_1(t)$  be the first eigenvalue of  $\omega_\varphi$  along the Kähler-Ricci flow. Then for any constants  $\Lambda, \sigma > 0$  there exists  $\epsilon_0(\Lambda, \sigma) > 0$  small enough such that if for all  $t \in [0, T]$*

$$Ca(\omega_\varphi) < \epsilon_0, \quad |Ric|(\omega_\varphi) \leq \Lambda, \quad C_S(\omega_\varphi) < \sigma, \quad (4.11)$$

*we have*

$$\lambda_1(t) \geq \frac{\lambda_1(0)}{1 + \lambda_1(0)(1 - e^{-3\epsilon t})} e^{-3\epsilon t}, \quad \forall t \in [0, T],$$

*where  $\epsilon = 3C(\Lambda, n)\sigma\epsilon_0^{\frac{1}{n}}$ .*

*Proof.* It follows from (3.2) that

$$\frac{d\lambda_1(t)}{dt} = \int_M (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) \omega_\varphi^n - \int_M R|\nabla f|^2 \omega_\varphi^n + \lambda_1 \int_M Rf^2 \omega_\varphi^n. \quad (4.12)$$

Recall the Sobolev inequality:

$$\left( \int_M |f|^p \omega_\varphi^n \right)^{\frac{2}{p}} \leq C_S \int_M (|\nabla f|^2 + f^2) \omega_\varphi^n, \quad (4.13)$$

where  $p = \frac{2n}{n-1}$ . Now we calculate :

$$\begin{aligned} \int_M (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) \omega_\varphi^n &\geq - \left( \int_M |Ric(\omega_\varphi) - \omega_\varphi|^n \right)^{\frac{1}{n}} \cdot \left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} \\ &\geq -C(\Lambda, n) Ca(\omega_\varphi)^{\frac{1}{n}} \cdot \left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} \end{aligned}$$

where the last step follows from the assumption on Ricci curvature. Now using the Sobolev inequality (4.13) we get:

$$\begin{aligned} \left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} &\leq C_S \left( \int_M |\nabla^2 f|^2 + \int_M |\nabla f|^2 \right) \\ &= C_S \left( \int_M |f_{i\bar{j}}|^2 + \int_M |f_{i\bar{j}}|^2 + \int_M |\nabla f|^2 \right) \\ &= C_S \left( \int_M |f_{i\bar{j}}|^2 + \int_M Ric(\omega_\varphi)(\nabla f, \nabla f) + \int_M |f_{i\bar{j}}|^2 + \int_M |\nabla f|^2 \right) \\ &= C_S \left( 2 \int_M |f_{i\bar{j}}|^2 + \int_M (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) + 2 \int_M |\nabla f|^2 \right) \\ &\leq 2C_S \int_M \left( (\Delta f)^2 + |\nabla f|^2 \right) + C_S C(\Lambda, n) Ca(\omega_\varphi)^{\frac{1}{n}} \left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} \end{aligned}$$

If the Calabi energy is sufficiently small, this implies that

$$\left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} \leq 3C_S \int_M \left( (\Delta f)^2 + |\nabla f|^2 \right) = 3C_S(\lambda_1^2 + \lambda_1).$$

Hence, we have the estimate:

$$\int_M (Ric(\omega_\varphi) - \omega_\varphi)(\nabla f, \nabla f) \omega_\varphi^n \geq -\epsilon(\lambda_1^2 + \lambda_1), \quad (4.14)$$

where  $\epsilon = 3C(\Lambda, n)\sigma\epsilon_0^{\frac{1}{n}}$ . Similarly, we have the estimates

$$- \int_M R|\nabla f|^2 = \int_M (n - R)|\nabla f|^2 - n \int_M |\nabla f|^2 \geq -\epsilon(\lambda_1^2 + \lambda_1) - n\lambda_1, \quad (4.15)$$

and

$$\lambda_1 \int_M Rf^2 = \lambda_1 \int_M (R - n)f^2 + n\lambda_1 \geq -\epsilon(\lambda_1 + 1)\lambda_1 + n\lambda_1. \quad (4.16)$$

Combining (4.14)-(4.16) with (4.12), we get

$$\frac{d\lambda_1(t)}{dt} \geq -3\epsilon(\lambda_1(t) + 1)\lambda_1.$$

Applying the maximum principle, we get

$$\lambda_1(t) \geq \frac{\lambda_1(0)}{1 + \lambda_1(0)(1 - e^{-3\epsilon t})} e^{-3\epsilon t}.$$

□

Now using Lemma 4.6, we have the estimate

$$\lambda_1(T) \geq \frac{\lambda_1(0)}{1 + \lambda_1(0)(1 - e^{-3\epsilon' T})} e^{-3\epsilon' T}, \quad \epsilon' = 3C(\Lambda, n)\sigma(C(\Lambda, T)\epsilon^{\frac{1}{4}})^{\frac{1}{n}}.$$

The first eigenvalue  $\lambda_1(T) > 1 + \frac{\gamma}{2}$  if  $\epsilon$  is small enough. The proposition is established.

□

## 4.2 The iteration argument

Thanks to Proposition 4.1, we only need to prove the convergence of Kähler-Ricci flow under the assumption that the traceless Ricci curvature is sufficiently small. In other words, Theorem 1.8 follows immediately from the following:

**Theorem 4.7.** *Let  $(M, J)$  be a Kähler manifold with  $c_1(M) > 0$ . For any  $\gamma, \Lambda > 0$ , there exists a small positive constant  $\epsilon(\gamma, \Lambda) > 0$  such that for any metric  $g$  in the subspace of Kähler metrics*

$$\{ \omega_g \in 2\pi c_1(M) \mid \lambda_1(\omega_g) > 1 + \gamma, \quad |Rm|(\omega_g) \leq \Lambda, \quad |Ric(\omega_g) - \omega_g| \leq \epsilon \}, \quad (4.17)$$

*the Kähler-Ricci flow with the initial metric  $\omega_g$  will converge exponentially fast to a Kähler-Einstein metric.*

*Proof.* The proof consists of several parts.

**STEP 1.** In this step we give estimates near the initial time. Consider the Kähler-Ricci flow (2.2) with the normalization condition (2.3) and the assumption (4.17). By Lemma 2.1, the maximum principle implies that there exists  $T_1(\Lambda) > 0$  such that

$$|Rm|(t) \leq 2\Lambda, \quad |Ric - \omega|(t) \leq 2\epsilon, \quad \forall t \in [0, T_1]. \quad (4.18)$$

By Lemma 3.1, for sufficiently small  $\epsilon$  we have the estimate

$$\lambda_1(t) \geq \lambda_1(0)e^{-6n\epsilon t} \geq (1 + \gamma)e^{-6n\epsilon T_1} \geq 1 + \frac{2\gamma}{3}, \quad \forall t \in [0, T_1].$$

Since we have the eigenvalue estimate, by Theorem 3.6 the traceless Ricci curvature decays exponentially

$$|Ric - \omega|(t) \leq 2C_2(2\Lambda, \frac{2}{3}\gamma, \tau)\epsilon e^{-\alpha t}, \quad \forall t \in [\tau, T_1] \quad (4.19)$$

where  $\alpha = \frac{\gamma}{6}$ . Since the traceless Ricci curvature is small for  $t \in [0, \tau]$  by (4.18), we can assume (4.19) holds for all  $t \in [0, T_1]$ . Now we choose  $\epsilon$  sufficiently small such that

$$2C_2(2\Lambda, \frac{2}{3}\gamma, \tau)\epsilon < 1.$$

By Theorem 3.8, the full curvature tensor has the estimate

$$|Rm|(t) \leq \Lambda_0(1, \frac{\gamma}{6}), \quad t \in [0, T_1].$$

Set  $\Lambda_1 := \max\{2\Lambda, \Lambda_0(1, \frac{\gamma}{8})\}$  and  $H_1 = 2C_2(\Lambda_1, \frac{2}{3}\gamma, \tau)$ . We choose  $\epsilon$  small such that  $H_1\epsilon < \frac{1}{2}$ . By (4.19) we have

$$|Ric - \omega|(t) \leq H_1\epsilon e^{-\alpha t}, \quad \forall t \in [0, T_1].$$

**STEP 2.** Recall that in step 1 we have the estimates at time  $t = T_1$ ,

$$|Rm|(T_1) \leq \Lambda_1, \quad |Ric - \omega|(T_1) \leq H_1\epsilon e^{-\alpha T_1}, \quad \lambda_1(T_1) \geq 1 + \frac{2\gamma}{3}. \quad (4.20)$$

By Lemma 2.1, the maximum principle implies that there exists  $T_2(\Lambda_1) > 0$  such that

$$|Rm|(t) \leq 2\Lambda_1, \quad |Ric - \omega|(t) \leq 2H_1\epsilon e^{-\alpha T_1}, \quad t \in [T_1, T_1 + T_2]. \quad (4.21)$$

By Lemma 3.1 for sufficiently small  $\epsilon$  the first eigenvalue has the estimate

$$\lambda_1(t) \geq \lambda_1(T_1)e^{-3nH_1\epsilon T_2} \geq 1 + \frac{\gamma}{2}, \quad \forall t \in [T_1, T_1 + T_2]. \quad (4.22)$$

Then by (3.16) the Calabi energy decays exponentially for all time  $t \in [0, T_1 + T_2]$ :

$$Ca(t) \leq e^{-\frac{\gamma}{2}t} Ca(0) = H_1^2 \epsilon^2 e^{-\frac{\gamma}{2}t}, \quad t \in [0, T_1 + T_2]. \quad (4.23)$$

As in the proof of Theorem 3.6, the parabolic Moser iteration implies the traceless Ricci curvature

$$\begin{aligned} |Ric - \omega|(t) &\leq C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) \left( \int_{t-\tau}^t \int_M |Ric - \omega|^4(s) \omega_\varphi^n \wedge ds \right)^{\frac{1}{4}} \\ &\leq C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) H_1 \epsilon e^{-\alpha \frac{T_1}{2}} e^{-\frac{\gamma}{8}t}, \quad t \in [T_1, T_1 + T_2], \end{aligned}$$

where we have used the estimates (4.21) and (4.23). Set  $H_2 = \max\{H_1, C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) H_1 e^{-\alpha \frac{T_1}{2}}\}$ . Then we get the inequality

$$|Ric - \omega|(t) \leq H_2 \epsilon e^{-\frac{\gamma}{8}t}, \quad t \in [0, T_1 + T_2]. \quad (4.24)$$

Therefore, by Theorem 3.8 and the definition of  $\Lambda_1$ , we can choose  $\epsilon$  small such that  $H_2\epsilon < 1$  and

$$|Rm|(t) \leq \Lambda_1, \quad t \in [0, T_1 + T_2]. \quad (4.25)$$

**STEP 3.** Following the argument in STEP 2, we have the following lemma:

**Lemma 4.8.** *Suppose for some  $k \in \mathbb{N}$  the following estimates hold:*

$$|Rm|(t) \leq \Lambda_1, \quad |Ric - \omega|(t) \leq H_k \epsilon e^{-\frac{\gamma}{8}t}, \quad \lambda_1(t) \geq 1 + \frac{\gamma}{2}, \quad t \in [0, T_1 + (k-1)T_2],$$

*then there exists  $k_0 \in \mathbb{N}$  such that if  $k \geq k_0$  and  $\epsilon$  is sufficiently small (depending on  $k_0$ ), the above estimates still hold for all  $t \in [T_1 + (k-1)T_2, T_1 + kT_2]$ .*

*Proof.* The maximum principle and the definition of  $T_2$  imply that

$$|Rm|(t) \leq 2\Lambda_1, \quad |Ric - \omega|(t) \leq 2H_k \epsilon e^{-\frac{\gamma}{8}(T_1 + (k-1)T_2)}, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2]. \quad (4.26)$$

Then by Lemma 3.1 the first eigenvalue

$$\begin{aligned} \lambda(t) &\geq \lambda(T_1 + (k-1)T_2) e^{-3nH_k \epsilon T_2} \\ &\geq \lambda(0) e^{-\frac{24n}{\gamma} H_k \epsilon} e^{-3nH_k \epsilon T_2} \\ &\geq 1 + \frac{\gamma}{2}, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2], \end{aligned}$$

if we choose  $\epsilon$  sufficiently small (depending on  $H_k$ ). Here we have used the inequality

$$\lambda(T_1 + (k-1)T_2) \geq \lambda_1(0) e^{-\frac{24n}{\gamma} H_k \epsilon (1 - e^{-\frac{\gamma}{8}t})} \geq \lambda(0) e^{-\frac{24n}{\gamma} H_k \epsilon}.$$

Then by (3.16) the Calabi energy decays exponentially

$$Ca(t) \leq e^{-\frac{\gamma}{2}t} Ca(0) = H_k^2 \epsilon^2 e^{-\frac{\gamma}{2}t}, \quad t \in [0, T_1 + kT_2], \quad (4.27)$$

Combining (4.26) with (4.27), we can estimate the traceless Ricci curvature by the parabolic Moser iteration as in step 2,

$$\begin{aligned} |Ric - \omega|(t) &\leq C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) \left( \int_{t-\tau}^t \int_M |Ric - \omega|^4(s) \omega_\varphi^n \wedge ds \right)^{\frac{1}{4}} \\ &\leq H_{k+1} \epsilon e^{-\frac{\gamma}{8}t}, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2], \end{aligned}$$

where  $H_{k+1}$  is defined by

$$H_{k+1} = H_k \cdot C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) \left( 2e^{-\frac{\gamma}{8}(T_1 + (k-1)T_2)} \right)^{\frac{1}{2}} \left( \frac{2}{\gamma} e^{\frac{\gamma}{2}\tau} \right)^{\frac{1}{4}}.$$

Then there exists  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$

$$C_2(2\Lambda_1, \frac{\gamma}{2}, \tau) \left( 2e^{-\frac{\gamma}{8}(T_1 + (k-1)T_2)} \right)^{\frac{1}{2}} \left( \frac{2}{\gamma} e^{\frac{\gamma}{2}\tau} \right)^{\frac{1}{4}} \leq 1.$$

Hence, we have the estimate for  $k \geq k_0$ ,

$$|Ric - \omega|(t) \leq H_{k_0} \epsilon e^{-\frac{\gamma}{8}t}, \quad t \in [0, T_1 + kT_2]. \quad (4.28)$$

We choose  $\epsilon$  small such that  $H_{k_0} \epsilon < 1$ . Therefore, by Theorem 3.8 and the definition of  $\Lambda_1$  we have

$$|Rm|(t) \leq \Lambda_1, \quad t \in [0, T_1 + kT_2].$$

The lemma is proved. □



**STEP 4.** By Lemma 4.8, the bisectional curvature is uniformly bounded and the traceless Ricci curvature decays exponentially, and therefore the  $W^{1,2}$  norm of  $\frac{\partial \varphi}{\partial t} - c(t)$  decays exponentially. Then following the argument in [8], the Kähler-Ricci flow will converge exponentially fast to a Kähler-Einstein metric. This theorem is proved.  $\square$

## 5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The key observation is that if a Kähler metric  $g$  with possibly different complex structure  $J$  is sufficiently close to a Kähler-Einstein metric  $(g_{KE}, J_{KE})$ , the conditions in Theorem 1.8 or Theorem 4.7 are automatically satisfied, and Theorem 1.4 follows immediately.

To verify the condition on the first eigenvalue, we need the following result. Let  $g$  be a Riemannian metric on an  $2n$ -dimensional Riemannian manifold with the first eigenvalue  $\lambda_1(g)$ . Define the space of Riemannian metrics

$$\mathcal{A}_\delta = \{g' \mid (1 - \delta)g \leq g' \leq (1 + \delta)g\}$$

for small  $\delta > 0$ . We have the following lemma

**Lemma 5.1.** *For any  $\epsilon > 0$ , there exists a  $\delta_0(\epsilon) > 0$  such that for any  $h \in \mathcal{A}_\delta$  with  $0 < \delta < \delta_0$ , the first eigenvalue of  $h$  satisfies*

$$\lambda_1(h) > \lambda_1(g)(1 - \epsilon).$$

*Proof.* In fact, for any smooth function  $f$  with  $\int_M f dV_h = 0$ , we have

$$\begin{aligned} \frac{1}{V_h} \int_M |\nabla f|_h^2 dV_h &\geq \frac{(1 - \delta)^n}{1 + \delta} \frac{1}{V_h} \int_M |\nabla f|_g^2 dV_g \\ &\geq \lambda_1(g) \frac{(1 - \delta)^n}{1 + \delta} \frac{V_g}{V_h} \left( \frac{1}{V_g} \int_M f^2 dV_g - \left( \frac{1}{V_g} \int_M f dV_g \right)^2 \right). \end{aligned}$$

Notice that

$$\left( \int_M f dV_g \right)^2 \leq \left( \int_M f dV_h \right)^2 + \left( \int_M f \left( \frac{dV_g}{dV_h} - 1 \right) dV_h \right)^2.$$

By the assumption,

$$\frac{1}{(1 + \delta)^n} - 1 \leq \frac{dV_g}{dV_h} - 1 \leq \frac{1}{(1 - \delta)^n} - 1,$$

therefore,

$$\left( \int_M f dV_g \right)^2 \leq C_\delta^2 V_h \int_M f^2 dV_h$$

where  $C_\delta = \max\{\frac{1}{(1-\delta)^n} - 1, 1 - \frac{1}{(1+\delta)^n}\}$ . Then

$$\begin{aligned}
& \frac{1}{V_h} \int_M |\nabla f|_h^2 dV_h \\
& \geq \lambda_1(g) \frac{(1-\delta)^n}{1+\delta} \frac{V_g}{V_h} \left( \frac{1}{V_g} \frac{1}{(1+\delta)^n} \int_M f^2 dV_h - \left( \frac{1}{V_g} \int_M f dV_g \right)^2 \right) \\
& \geq \lambda_1(g) \frac{(1-\delta)^n}{1+\delta} \frac{V_g}{V_h} \left( \frac{V_h}{V_g} \frac{1}{(1+\delta)^n} - \frac{C_\delta^2 V_h^2}{V_g^2} \right) \frac{1}{V_h} \int_M f^2 dV_h \\
& \geq \lambda_1(g) (1-\epsilon) \frac{1}{V_h} \int_M f^2 dV_h.
\end{aligned}$$

This implies

$$\lambda_1(h) \geq \lambda_1(g)(1-\epsilon).$$

□

*Proof of Theorem 1.4.* Let  $(g, J)$  be a Kähler metric with

$$\|(g, J) - (g_{KE}, J_{KE})\|_{C^2} \leq \epsilon, \quad (5.1)$$

for sufficiently small  $\epsilon > 0$ , we need to check that  $(g, J)$  satisfies the assumption (4.17) of Theorem 4.7. In fact, since  $(g_{KE}, J_{KE})$  has no holomorphic vector fields, the first eigenvalue of the Laplacian  $\Delta_{KE}$  is strictly greater than 1, and by Lemma 5.1  $\lambda_1(g) \geq \lambda_1(g_{KE})(1-\epsilon) > 1$ . The rest of the assumption (4.17) can be easily verified by (5.1)(cf. Lemma 2.7, 2.8 in [15]). The theorem is proved.

□

## 6 Proof of Theorem 1.6 and 1.10

In this section, we will use the pre-stable condition to drop the assumptions that  $M$  has no nonzero holomorphic vector fields, and the dependence of the initial first eigenvalue of the Laplacian. The idea of the proof is similar to that in section 4.

First, we will prove Theorem 1.10. As in Proposition 4.1 we prove that the traceless Ricci curvature is small along the Kähler-Ricci flow after a short time.

**Proposition 6.1.** *Given a Kähler metric  $\omega_g$  satisfying the properties*

$$|Rm|(\omega_g) \leq \Lambda, \quad Ca(\omega_g) < \epsilon, \quad (6.1)$$

*for some constants  $\Lambda > 0$  and sufficiently small  $\epsilon > 0$ . Then for the solution  $\omega(t)$  of the Kähler-Ricci flow with the initial metric  $\omega_g$ , there exists  $T(\Lambda)$  and  $\epsilon_0(\Lambda) > 0$  such that if  $\epsilon \in (0, \epsilon_0)$  then at time  $T$  the metric  $\omega(T)$  satisfies*

$$|Rm|(T) \leq 2\Lambda, \quad |Ric - \omega|(T) \leq C\epsilon^{\frac{1}{4}}, \quad (6.2)$$

*for some constant  $C(\Lambda)$ .*

We see that this proposition follows immediately from the proof of Proposition 4.1. Hence, Theorem 1.10 is a direct corollary of the following result:

**Theorem 6.2.** *Suppose  $(M, J)$  is pre-stable and the Futaki invariant of the class  $2\pi c_1(M)$  vanishes. For any  $\Lambda > 0$ , there exists  $\epsilon(\Lambda) > 0$  such that for any metric  $g$  with its Kähler form  $\omega_g$  in the subspace of Kähler metrics*

$$\{ \omega_g \in 2\pi c_1(M) \mid |Rm|(\omega_g) \leq \Lambda, \quad |Ric(\omega_g) - \omega_g| \leq \epsilon \}, \quad (6.3)$$

*the Kähler Ricci flow with the initial metric  $\omega_g$  will converge exponentially fast to a Kähler-Einstein metric.*

*Proof.* We follow closely the proof of Theorem 4.7.

**STEP 1.** In this step we give estimates near the initial time. Consider the Kähler-Ricci flow (2.2) with the normalization condition (2.3) and the assumption (6.3). By Lemma 2.1, the maximum principle implies that there exists  $T_1(\Lambda) > 0$  such that

$$|Rm|(t) \leq 2\Lambda, \quad |Ric - \omega|(t) \leq 2\epsilon, \quad \forall t \in [0, T_1]. \quad (6.4)$$

By Shi's estimates(cf. [29]), all the derivatives of the curvature tensor are uniformly bounded for time  $t \in [\frac{\tau}{2}, T_1]$ . Then applying Theorem 3.5, if  $\epsilon$  is sufficiently small there exists  $\gamma_0 := \gamma(2\epsilon, 2\Lambda) > 0$  for time  $t \in [\frac{\tau}{2}, T_1]$  such that the inequalities (3.7) and (3.8) hold. Since we have the eigenvalue estimate, by Theorem 3.7 the traceless Ricci curvature decays exponentially

$$|Ric - \omega|(t) \leq 2C_3(2\Lambda, \frac{\gamma_0}{4}, \tau)\epsilon e^{-\frac{\gamma_0}{4}t}, \quad \forall t \in [\tau, T_1]. \quad (6.5)$$

Since the traceless Ricci curvature is small for  $t \in [0, \tau]$  by (6.4), we can assume (6.5) holds and also  $\gamma \geq \gamma_0$  for all  $t \in [0, T_1]$ . Now we choose  $\epsilon$  sufficiently small such that

$$2C_3(2\Lambda, \frac{\gamma_0}{4}, \tau)\epsilon < 1.$$

By Theorem 3.8, the full curvature tensor has the estimate

$$|Rm|(t) \leq \Lambda_0(1, \frac{\gamma_0}{4}), \quad t \in [0, T_1].$$

Set  $\Lambda_1 := \max\{2\Lambda, \Lambda_0(1, \frac{\gamma_0}{8})\}$  and  $H_1 = 2C_3(2\Lambda, \frac{\gamma_0}{4}, \tau)$ . We choose  $\epsilon$  small such that  $H_1\epsilon < \frac{1}{2}$ . By (6.5) we have

$$|Ric - \omega|(t) \leq H_1\epsilon e^{-\frac{\gamma_0}{4}t}, \quad \forall t \in [0, T_1].$$

**STEP 2.** Recall that in step 1 we have the estimates at time  $t = T_1$ ,

$$|Rm|(T_1) \leq \Lambda_1, \quad |Ric - \omega|(T_1) \leq H_1\epsilon e^{-\frac{\gamma_0}{4}T_1}. \quad (6.6)$$

By Lemma 2.1, the maximum principle implies that there exists  $T_2(\Lambda_1) > 0$  such that

$$|Rm|(t) \leq 2\Lambda_1, \quad |Ric - \omega|(t) \leq 2H_1\epsilon e^{-\frac{\gamma_0}{4}T_1}, \quad t \in [T_1, T_1 + T_2].$$

By Theorem 3.5, we can choose  $\epsilon$  sufficiently small such that  $\gamma(2H_1\epsilon, 2\Lambda_1) \geq \gamma_1$  for some constant  $\gamma_1 \in (0, \gamma_0)$ . Then by (3.16) the Calabi energy decays exponentially for all time  $t \in [0, T_1 + T_2]$ :

$$Ca(t) \leq e^{-\gamma_1 t} Ca(0) = H_1^2 \epsilon^2 e^{-\gamma_1 t}, \quad t \in [0, T_1 + T_2]. \quad (6.7)$$

As in the proof of Theorem 3.6, the parabolic Moser iteration implies the traceless Ricci curvature

$$\begin{aligned} |Ric - \omega|(t) &\leq C_3(2\Lambda_1, \gamma_1, \tau) \left( \int_{t-\tau}^t \int_M |Ric - \omega|^4(s) \omega_\varphi^n \wedge ds \right)^{\frac{1}{4}} \\ &\leq C_3(2\Lambda_1, \gamma_1, \tau) H_1 \epsilon e^{-\frac{\gamma_0 T_1}{8}} e^{-\frac{\gamma_1}{4} t}, \quad t \in [T_1, T_1 + T_2]. \end{aligned}$$

Set  $H_2 = \max\{H_1, C_3(2\Lambda_1, \gamma_1, \tau) H_1 e^{-\frac{\gamma_0 T_1}{8}}\}$ , we get the inequality

$$|Ric - \omega|(t) \leq H_2 \epsilon e^{-\frac{\gamma_1}{4} t}, \quad t \in [0, T_1 + T_2]. \quad (6.8)$$

Therefore, by Theorem 3.8 and the definition of  $\Lambda_1$ , we can choose  $\epsilon$  small such that  $H_2 \epsilon < 1$  and

$$|Rm|(t) \leq \Lambda_1, \quad t \in [0, T_1 + T_2]. \quad (6.9)$$

**STEP 3.** Following the argument in STEP 2, we have the following lemma:

**Lemma 6.3.** *Suppose for some  $k \in \mathbb{N}$  the following estimates hold:*

$$|Rm|(t) \leq \Lambda_1, \quad |Ric - \omega|(t) \leq H_k \epsilon e^{-\frac{\gamma_1}{4} t}, \quad \forall t \in [0, T_1 + (k-1)T_2],$$

*then there exists  $k_0 \in \mathbb{N}$  such that if  $k \geq k_0$  and  $\epsilon$  is sufficiently small (depending on  $k_0$ ), the above estimates still hold for all  $t \in [T_1 + (k-1)T_2, T_1 + kT_2]$ .*

*Proof.* The maximum principle and the definition of  $T_2$  imply that

$$|Rm|(t) \leq 2\Lambda_1, \quad |Ric - \omega|(t) \leq 2H_k \epsilon e^{-\frac{\gamma_1}{4}(T_1 + (k-1)T_2)}, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2]. \quad (6.10)$$

Then if  $\epsilon$  is sufficiently small (depending on  $H_k$ ), the constant  $\gamma$  in Theorem 3.5 satisfies

$$\gamma(2\Lambda_1, 2H_k) \geq \gamma_1 > 0, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2].$$

Then by (3.16) the Calabi energy decays exponentially

$$Ca(t) \leq e^{-\gamma_1 t} Ca(0) = H_k^2 \epsilon^2 e^{-\gamma_1 t}, \quad t \in [0, T_1 + kT_2]. \quad (6.11)$$

Combining (6.10) with (6.11), we can estimate the traceless Ricci curvature by the parabolic Moser iteration as in step 2,

$$\begin{aligned} |Ric - \omega|(t) &\leq C_3(2\Lambda_1, \gamma_1, \tau) \left( \int_{t-\tau}^t \int_M |Ric - \omega|^4(s) \omega_\varphi^n \wedge ds \right)^{\frac{1}{4}} \\ &\leq H_{k+1} \epsilon e^{-\frac{\gamma_1}{4} t}, \quad t \in [T_1 + (k-1)T_2, T_1 + kT_2], \end{aligned}$$

where  $H_{k+1}$  is defined by

$$H_{k+1} = H_k \cdot C_3(2\Lambda_1, \gamma_1, \tau) \left( 2e^{-\frac{\gamma_1}{4}(T_1+(k-1)T_2)} \right)^{\frac{1}{2}} \left( \frac{1}{\gamma_1} e^{\gamma_1 \tau} \right)^{\frac{1}{4}}.$$

Then there exists  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$

$$C_3(2\Lambda_1, \gamma_1, \tau) \left( 2e^{-\frac{\gamma_1}{8}(T_1+(k-1)T_2)} \right)^{\frac{1}{2}} \left( \frac{1}{\gamma_1} e^{\gamma_1 \tau} \right)^{\frac{1}{4}} \leq 1.$$

Hence, we have the estimate for  $k \geq k_0$ ,

$$|Ric - \omega|(t) \leq H_{k_0} \epsilon e^{-\frac{\gamma_1}{4}t}, \quad t \in [0, T_1 + kT_2]. \quad (6.12)$$

We choose  $\epsilon$  small such that  $H_{k_0} \epsilon < 1$ . Therefore, by Theorem 3.8 and the definition of  $\Lambda_1$  we have

$$|Rm|(t) \leq \Lambda_1, \quad t \in [0, T_1 + kT_2].$$

The lemma is proved. □

**STEP 4.** By Lemma 6.3, the bisectional curvature is uniformly bounded and the traceless Ricci curvature decays exponentially, and therefore the  $W^{1,2}$  norm of  $\frac{\partial \varphi}{\partial t} - c(t)$  decays exponentially. Then following the argument in [8], the Kähler-Ricci flow will converge exponentially fast to a Kähler-Einstein metric. This theorem is proved. □

*Proof of Theorem 1.6.* Theorem 1.6 is a direct corollary of Theorem 6.2. □

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